
Calculus

Concepts, Computation & Applications

An AP Calculus AB & BC Companion

Derivatives

Student Edition

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How to Use This Chapter

This chapter was developed from the sequence of derivative problems used in AP Calculus class: derivative from first principles, core differentiation rules, product and quotient rules, the chain rule, tangent-line and instantaneous-rate applications, implicit differentiation, table-based derivatives, motion, and cumulative applications.

The goal is not merely to memorize formulas. Students should learn to recognize *what a derivative means*, select an efficient rule, organize algebra carefully, and interpret the result in context.

AP Connection

Worked examples are part of the notes and are intended to be studied. Practice problems are left unsolved so they can be completed in class, for homework, or as review.

UNIT 3

Differentiation

Learning Goals

By the end of this chapter, you should be able to:

- interpret a derivative as a limit, tangent slope, and instantaneous rate of change;
- differentiate algebraic, trigonometric, exponential, and logarithmic functions;
- apply the product, quotient, and chain rules, including nested compositions;
- differentiate implicitly defined relations;
- compute derivatives from tables of function values and derivative values;
- analyze tangent lines, velocity, acceleration, and other rate-of-change applications.

3.1 The Derivative from First Principles

Definition — Derivative at a Point

Let f be defined on an interval containing a . The derivative of f at $x = a$ is

$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h},$$

provided the limit exists.

The quotient

$$\frac{f(a+h) - f(a)}{h}$$

is the slope of a secant line. As $h \rightarrow 0$, the second point approaches the first, and the secant slope approaches the slope of the tangent line.

Definition — Derivative Function

When the derivative exists for values of x in an interval,

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}.$$

The derivative function records the instantaneous rate of change of f at each input x .

Worked Example 3.1

Evaluate

$$\lim_{h \rightarrow 0} \frac{(3+h)^2 - 9}{h}$$

and explain its meaning for $f(x) = x^2$.

Solution.

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{(3+h)^2 - 9}{h} &= \lim_{h \rightarrow 0} \frac{9 + 6h + h^2 - 9}{h} \\ &= \lim_{h \rightarrow 0} (6 + h) = 6. \end{aligned}$$

This is $f'(3)$, so the tangent line to $y = x^2$ at $x = 3$ has slope 6.

Worked Example 3.2

Use the definition of the derivative to find $f'(x)$ for

$$f(x) = \sqrt{x+4}.$$

Solution.

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{\sqrt{x+h+4} - \sqrt{x+4}}{h} \\ &= \lim_{h \rightarrow 0} \frac{\sqrt{x+h+4} - \sqrt{x+4}}{h} \cdot \frac{\sqrt{x+h+4} + \sqrt{x+4}}{\sqrt{x+h+4} + \sqrt{x+4}} \\ &= \lim_{h \rightarrow 0} \frac{1}{\sqrt{x+h+4} + \sqrt{x+4}} \\ &= \frac{1}{2\sqrt{x+4}}. \end{aligned}$$

AP Connection

A limit of the form

$$\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

should immediately be recognized as a derivative. On AP questions, you may be asked to identify the derivative, evaluate it, or interpret what it means in context.

Practice: First Principles

Practice 3.1.1. Evaluate

$$\lim_{h \rightarrow 0} \frac{\ln(x+h) - \ln x}{h}, \quad x > 0.$$

Practice 3.1.2. Evaluate

$$\lim_{h \rightarrow 0} \frac{e^{x+h} - e^x}{h}.$$

Practice 3.1.3. Let $f(x) = 4x^3$. Find $f'(x)$ using the definition of the derivative and show the complete limit process.

3.2 Core Differentiation Rules

Once the derivative has been defined as a limit, derivative rules allow us to calculate efficiently without rebuilding the limit each time.

Power, Constant, and Sum Rules

For differentiable functions and constants c :

$$\frac{d}{dx}(c) = 0, \quad \frac{d}{dx}(x^n) = nx^{n-1},$$

and

$$\frac{d}{dx}[f(x) \pm g(x)] = f'(x) \pm g'(x).$$

Also,

$$\frac{d}{dx}[cf(x)] = cf'(x).$$

Useful Basic Derivatives

$$\frac{d}{dx}(e^x) = e^x, \quad \frac{d}{dx}(\ln x) = \frac{1}{x},$$

$$\frac{d}{dx}(\sin x) = \cos x, \quad \frac{d}{dx}(\cos x) = -\sin x,$$

$$\frac{d}{dx}(\tan x) = \sec^2 x, \quad \frac{d}{dx}(\cot x) = -\csc^2 x.$$

Worked Example 3.3

Differentiate

$$f(x) = 6x^5 - 4x^3 + 7x^2 - 9x + 12.$$

Solution.

$$f'(x) = 30x^4 - 12x^2 + 14x - 9.$$

Worked Example 3.4

Differentiate

$$f(x) = x^3 \cos x + 4e^x - 5 \ln x + 7\sqrt{x}.$$

Solution. The first term requires the product rule, which will be formalized in the next section. Using it here,

$$f'(x) = 3x^2 \cos x - x^3 \sin x + 4e^x - \frac{5}{x} + \frac{7}{2\sqrt{x}}.$$

Common Mistake

Do not lower the exponent without multiplying by the old exponent. For example,

$$\frac{d}{dx}(x^5) \neq x^4; \quad \frac{d}{dx}(x^5) = 5x^4.$$

Practice: Basic Rules**Practice 3.2.1.** Find $f'(x)$ if

$$f(x) = 4x^5 - 3x^3 + 7x^2 - 5x + 8.$$

Practice 3.2.2. Find $\frac{d}{dx}(\cot x)$ by writing $\cot x = \frac{\cos x}{\sin x}$ and using the quotient rule.

3.3 Product and Quotient Rules

Product Rule

If f and g are differentiable, then

$$(fg)' = f'g + fg'.$$

Quotient Rule

If $g(x) \neq 0$, then

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}.$$

Worked Example 3.5

Differentiate

$$g(x) = (x^3 - 2x)(4x^2 + 5).$$

Solution.

$$g'(x) = (3x^2 - 2)(4x^2 + 5) + (x^3 - 2x)(8x).$$

This form is already correct; expansion is optional unless the problem asks for it.

Worked Example 3.6

Differentiate

$$p(x) = \frac{3x^2 - 1}{x^3 + 2}.$$

Solution.

$$p'(x) = \frac{(6x)(x^3 + 2) - (3x^2 - 1)(3x^2)}{(x^3 + 2)^2}.$$

AP Connection

AP questions often reward a correct derivative structure even when algebraic expansion is unnecessary. Unless simplification creates a clear advantage, keep product-rule and quotient-rule answers organized rather than expanding everything.

Practice: Product and Quotient Rules

Practice 3.3.1. Find $f'(x)$ if

$$f(x) = (x^3 - 2x) \sin x.$$

Practice 3.3.2. Find $f'(x)$ if

$$f(x) = \frac{x^3 + 4x - 2}{x^2 + 1}.$$

Practice 3.3.3. Find $f'(x)$ if

$$f(x) = e^x \ln x.$$

Practice 3.3.4. Find $f'(x)$ if

$$f(x) = \sqrt{x} \tan x.$$

Practice 3.3.5. Differentiate

$$f(x) = \frac{x^2 \sin x + \ln x}{x + 1}.$$

3.4 The Chain Rule

A composite function has one function acting inside another. The chain rule differentiates the outer function and multiplies by the derivative of the inner function.

Chain Rule

If $y = f(g(x))$, then

$$\frac{dy}{dx} = f'(g(x))g'(x).$$

A useful verbal pattern is:

derivative of the outside $\times$ derivative of the inside.

Worked Example 3.7

Differentiate

$$y = (3x^2 - 5x + 1)^4.$$

Solution. The outer function is u^4 and the inner function is $u = 3x^2 - 5x + 1$:

$$y' = 4(3x^2 - 5x + 1)^3(6x - 5).$$

Worked Example 3.8

Differentiate

$$y = e^{x^2 - 4x + 1}.$$

Solution.

$$y' = (2x - 4)e^{x^2 - 4x + 1}.$$

Worked Example 3.9 — Nested Chain Rule

Differentiate

$$k(x) = [(2x^2 - 3x + 4)^3 + 1]^5.$$

Solution.

$$k'(x) = 5[(2x^2 - 3x + 4)^3 + 1]^4 \cdot 3(2x^2 - 3x + 4)^2(4x - 3).$$

There are two nested layers, so the chain rule is applied twice.

Common Mistake

For $y = \sin(x^2 + 3x)$, the derivative is not merely $\cos(x^2 + 3x)$. The inner derivative must also appear:

$$y' = (2x + 3)\cos(x^2 + 3x).$$

Practice: Chain Rule

Practice 3.4.1. Differentiate

$$y = (2x^2 - 3x + 4)^4.$$

Then evaluate the slope at $x = 1$.

Practice 3.4.2. Let

$$f(x) = \sqrt{5x^3 + 1}.$$

Find the instantaneous rate of change at $x = 1$.

Practice 3.4.3. Differentiate

$$y = \sin(3x^2 + 2x).$$

Practice 3.4.4. Differentiate

$$y = \ln(2x^3 + 5x).$$

Practice 3.4.5. Suppose

$$h(x) = f(g(x))t(x),$$

where f , g , and t are differentiable. Find $h'(x)$.

3.5 Tangent Lines and Instantaneous Rates of Change

The derivative is not only a formula. Its value has geometric and physical meaning.

Definition — Tangent Slope

If f is differentiable at $x = a$, then the slope of the tangent line is

$$m = f'(a).$$

The tangent line through $(a, f(a))$ is

$$y - f(a) = f'(a)(x - a).$$

Definition — Instantaneous Rate of Change

If a quantity Q depends on a variable t , then $Q'(t)$ gives the instantaneous rate at which Q changes with respect to t . Units are

$$\frac{\text{units of } Q}{\text{units of } t}.$$

Worked Example 3.10

Let

$$f(x) = x^3 - 4x^2 + 2x + 1.$$

Find the tangent line at $x = 2$.

Solution.

$$f'(x) = 3x^2 - 8x + 2, \quad f'(2) = 12 - 16 + 2 = -2.$$

Also,

$$f(2) = 8 - 16 + 4 + 1 = -3.$$

Therefore

$$y + 3 = -2(x - 2),$$

or

$$\boxed{y = -2x + 1}.$$

Worked Example 3.11

The amount of water in a tank is

$$R(t) = 4t^3 - 15t^2 + 12t,$$

where R is in gallons and t is in hours. Find the instantaneous rate of change at $t = 2$.

Solution.

$$R'(t) = 12t^2 - 30t + 12,$$

so

$$\boxed{R'(2) = 0 \text{ gal/hr}}.$$

At that instant, the amount of water is momentarily neither increasing nor decreasing.

Practice: Tangent Slopes and Rates

Practice 3.5.1. Let

$$f(x) = 3x^4 - 5x^2 + 2x - 7.$$

Find the slope of the tangent line at $x = 1$.

Practice 3.5.2. Let

$$y = (x^2 + 3)(2x - 5).$$

Find the slope of the tangent line at $x = 2$.

Practice 3.5.3. Let

$$q(x) = \frac{x^2 + 1}{x + 2}.$$

Find the slope of the tangent line at $x = 1$.

Practice 3.5.4. Let

$$h(x) = \ln(3x^2 + 2x + 1).$$

Find the instantaneous rate of change at $x = 1$.

3.6 Motion: Position, Velocity, and Acceleration

Motion Relationships

If $s(t)$ is position, then

$$v(t) = s'(t)$$

is velocity, and

$$a(t) = v'(t) = s''(t)$$

is acceleration.

Velocity includes direction. A positive velocity indicates motion in the positive direction; a negative velocity indicates motion in the negative direction.

Worked Example 3.12

A particle has position

$$s(t) = 2t^3 - 9t^2 + 12t + 1.$$

Find its instantaneous velocity at $t = 3$.

Solution.

$$v(t) = s'(t) = 6t^2 - 18t + 12.$$

Therefore

$$v(3) = 12 \text{ m/s}.$$

Practice: Motion

Practice 3.6.1. A particle has position

$$s(t) = t^3 - 6t^2 + 9t + 4$$

meters. Find:

- (a) $v(t)$,
- (b) $v(2)$,
- (c) $a(t)$,
- (d) $a(2)$.

Interpret $v(2)$.

Practice 3.6.2. A ball is thrown upward from a platform. Its height is

$$h(t) = -16t^2 + 64t + 20$$

feet. Find the instantaneous velocity at $t = 1.5$ and determine whether the ball is moving upward or downward.

3.7 Implicit Differentiation

Some curves are given by equations in which y is not isolated. We can still differentiate both sides with respect to x , remembering that y is a function of x .

Key Idea

Whenever a derivative is taken of an expression involving y , multiply by $y' = \frac{dy}{dx}$ through the chain rule. For example,

$$\frac{d}{dx}(y^3) = 3y^2y'.$$

Worked Example 3.13

Given

$$x^2 + y^2 = 25,$$

find $\frac{dy}{dx}$ and the tangent slope at $(3, 4)$.

Solution.

$$2x + 2yy' = 0,$$

so

$$y' = -\frac{x}{y}.$$

At $(3, 4)$,

$$y' = -\frac{3}{4}.$$

Worked Example 3.14

Find $\frac{dy}{dx}$ if

$$x^2 + xy + y^2 = 7.$$

Solution.

$$2x + (xy' + y) + 2yy' = 0.$$

Collect the terms containing y' :

$$y'(x + 2y) = -(2x + y).$$

Therefore

$$y' = -\frac{2x + y}{x + 2y}.$$

Worked Example 3.15

Find $\frac{dy}{dx}$ if

$$\sin(xy) = x + y.$$

Solution.

$$\cos(xy)(xy' + y) = 1 + y'.$$

Collect y' terms:

$$xy' \cos(xy) - y' = 1 - y \cos(xy).$$

Thus

$$y' = \frac{1 - y \cos(xy)}{x \cos(xy) - 1}.$$

Common Mistake

In a term such as xy , both x and y vary with x . Therefore

$$\frac{d}{dx}(xy) = xy' + y,$$

not simply y and not simply xy' .

Practice: Implicit Differentiation

Practice 3.7.1. Find $\frac{dy}{dx}$ if

$$x^3 + y^3 = 9xy.$$

Practice 3.7.2. Find $\frac{dy}{dx}$ if

$$x^2y + y^2 = 8.$$

Practice 3.7.3. Find $\frac{dy}{dx}$ if

$$\sin x + \cos y = 1.$$

Practice 3.7.4. Find $\frac{dy}{dx}$ if

$$e^{x+y} = xy.$$

Practice 3.7.5. Find $\frac{dy}{dx}$ if

$$\ln(x + y) = xy.$$

Practice 3.7.6. Find $\frac{dy}{dx}$ if

$$e^{xy} + x^2y^3 = 6.$$

3.8 Derivatives from Tables

A table may give values of functions and their derivatives without giving formulas. The derivative rules work exactly the same way; the main task is to substitute the correct values at the correct inputs.

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
1	3	2	2	-1
2	5	-3	4	2
4	-1	6	9	3

Worked Example 3.16

If $h(x) = f(x)g(x)$, find $h'(1)$.

Solution.

$$h'(1) = f'(1)g(1) + f(1)g'(1) = 2(2) + 3(-1) = \boxed{1}.$$

Worked Example 3.17

If $h(x) = f(g(x))$, find $h'(1)$.

Solution.

$$h'(1) = f'(g(1))g'(1).$$

Since $g(1) = 2$,

$$h'(1) = f'(2)(-1) = (-3)(-1) = \boxed{3}.$$

Common Mistake

For a composition, do not substitute $x = 1$ into f' automatically. If $h(x) = f(g(x))$, then the derivative uses $f'(g(x))$. First find the inside value $g(1)$, then locate that input in the table.

Practice: Table-Based Derivatives

Use the table above.

Practice 3.8.1. If

$$h(x) = \frac{g(x)}{f(x)},$$

find $h'(2)$.

Practice 3.8.2. If

$$h(x) = [f(x)]^2 + g(x),$$

find the instantaneous rate of change at $x = 1$.

Practice 3.8.3. If

$$h(x) = \sqrt{g(x)} + f(x),$$

find the slope of the tangent line at $x = 4$.

3.9 Cumulative AP-Style Practice

The following problems intentionally mix skills. Before differentiating, identify the structure of the function and decide which rules are needed.

Practice 3.9.1. *Multiple rules* Find $q'(x)$ if

$$q(x) = x^2(x^3 + 1) - \frac{2x - 5}{x^2 + 4}.$$

Practice 3.9.2. *Chain rule and tangent slope* Let

$$y = \sin(x^2 + 3x).$$

Find the slope of the tangent line at $x = 1$.

Practice 3.9.3. *Implicit tangent slope* The variables x and y satisfy

$$xe^y + y = 1.$$

Find the slope of the tangent line at $(1, 0)$.

Practice 3.9.4. *Implicit tangent slope* The variables x and y satisfy

$$x^3 + y^3 = 6xy.$$

Find the slope of the tangent line at $(3, 3)$.

Practice 3.9.5. *Rate of change* The radius of a circular ripple is

$$r(t) = \sqrt{4t^2 + 1}$$

feet, and its area is $A = \pi r^2$. Find $\frac{dA}{dt}$ at $t = 2$.

Practice 3.9.6. *Full tangent-line equation* Let

$$f(x) = x^3 - 4x^2 + 2x + 1.$$

Find the slope of the tangent line at $x = 2$, then write the equation of the tangent line.

AP Connection

When a problem asks for a *slope*, the final answer is a number. When it asks for a *derivative*, the answer is usually a function. When it asks for a *tangent line*, both the derivative value and the point on the original function are needed.

3.10 Chapter Summary

Derivative definition $f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$

Power rule $\frac{d}{dx} x^n = nx^{n-1}$

Product rule $(fg)' = f'g + fg'$

Quotient rule $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$

Chain rule $\frac{d}{dx} f(g(x)) = f'(g(x))g'(x)$

Tangent slope $m = f'(a)$ at $x = a$

Velocity $v(t) = s'(t)$

Acceleration $a(t) = v'(t) = s''(t)$

Implicit differentiation Differentiate both sides; every derivative of y produces a factor of y'